Annular Shell Isolation Near Zeta Zeros

Real-Axis Simplex Brackets and a Finite-Range Diagnostic

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Abstract

We describe a finite-range diagnostic for local isolation near nontrivial zeros of the Riemann zeta function. The construction was motivated by a real-axis "simplex bracket" for $\zeta(s)$ when $s > 1$, in which the reciprocal triangular and tetrahedral totals,

$$\sum_{n\geq 1} \frac{1}{T_n} = 2, \qquad \sum_{n\geq 1} \frac{1}{\binom{n+2}{3}} = \frac{3}{2},$$

place $\zeta(2) = \pi^2/6$ inside the interval $(3/2, 2)$. This real-axis bracket does not locate zeros and does not extend directly to the critical line, where $\zeta(s)$ is complex. It does, however, introduce the chamber arithmetic that later reappears in the zero-shell diagnostic:

$$m_{\mathbb{R}}(s) = \left\lfloor \frac{1}{\zeta(s) - 1} \right\rfloor + 1, \qquad m_{\text{isolate}}(\rho) = \left\lfloor \frac{1}{|\zeta'(\rho)| h_\rho} \right\rfloor + 1.$$

For a simple nontrivial zero $\rho_j = 1/2 + i\gamma_j$, Taylor expansion gives

$$\zeta(s) = \zeta'(\rho_j)(s - \rho_j) + O\big((s - \rho_j)^2\big)\,.$$

Thus the zero-centered level shell $|\zeta(s)| = 1/m$ has local first-order radius

$$r_m(\rho_j) \approx \frac{1}{m\,|\zeta'(\rho_j)|}.$$

Comparing this radius to the nearest-neighbor half-gap

$$h_j = \frac{1}{2}\min(\gamma_j - \gamma_{j-1},\, \gamma_{j+1} - \gamma_j)$$

reduces local shell isolation to the product

$$P_j = |\zeta'(\rho_j)|\, h_j, \qquad m_{\text{isolate}}(\rho_j) = \left\lfloor \frac{1}{P_j} \right\rfloor + 1.$$

This is not a new zeta invariant; it is a compact finite-data coordinate obtained from the linearization and the local zero spacing.

Across a baseline block of 4519 zeros, a disjoint block of 1000 zeros, and three high-altitude sentinel bands of 250 zeros each near $T = 10^5, 10^6$, and $9 \cdot 10^6$, the derivative magnitude $|\zeta'(\rho_j)|$ and the local half-gap h_j are positively rank-correlated. The strongest isolation outliers are dominated by adjacent close-zero pairs, where small gap and depressed derivative scale jointly suppress P_j. The reported maxima are finite-range observations only; no global bound is claimed or expected.

1 Scope and Caution

This note has a narrow purpose. It does not introduce a zero-finding algorithm, does not claim a new invariant of $\zeta(s)$, and has no implication for the Riemann Hypothesis. It describes a diagnostic layer applied *after* zero ordinates have been located. The diagnostic asks:

> Given a known zero, how small must the zero-centered $|\zeta|$ shell be before it remains inside that zero's local half-gap neighborhood?

The answer is controlled, to first order, by the product

$$P_j = \left|\zeta'(\rho_j)\right| h_j.$$

The empirical contribution is the finite-range audit of this product across several blocks of zeros and the observation that the outlier tail is dominated by close-pair events.

2 Real-Axis Simplex Brackets

2.1 The reciprocal-simplex ladder

For $d \geq 2$, define the d-simplex figurate number

$$S_d(n) = \binom{n+d-1}{d}.$$

The reciprocal-simplex sum is elementary:

$$\sum_{n=1}^{\infty} \frac{1}{S_d(n)} = \sum_{n=1}^{\infty} \frac{1}{\binom{n+d-1}{d}} = \frac{d}{d-1}.$$

For $d = 2$, $S_2(n) = T_n = n(n+1)/2$ is triangular, and

$$\sum_{n\geq 1} \frac{1}{T_n} = 2.$$

For $d = 3$, $S_3(n) = \binom{n+2}{3}$ is tetrahedral, and

$$\sum_{n\geq 1} \frac{1}{S_3(n)} = \frac{3}{2}.$$

Proposition 1 (Reciprocal-simplex sum)**.** *For every integer $d \geq 2$,*

$$\sum_{n=1}^{\infty} \frac{1}{\binom{n+d-1}{d}} = \frac{d}{d-1}.$$

Proof. Using the beta integral,

$$\frac{1}{\binom{n+d-1}{d}} = d\,B(n,d) = d\int_0^1 x^{n-1}(1-x)^{d-1}\,dx.$$

Therefore

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{\binom{n+d-1}{d}} &= d\int_0^1 (1-x)^{d-1} \sum_{n=1}^{\infty} x^{n-1}\,dx \\ &= d\int_0^1 (1-x)^{d-2}\,dx = \frac{d}{d-1}.\end{aligned}$$

□

2.2 The triangular–tetrahedral bracket for $\zeta(2)$

The Basel value satisfies

$$\zeta(2) = \frac{\pi^2}{6} \approx 1.644934.$$

Thus

$$\frac{3}{2} < \zeta(2) < 2.$$

In this sense, $\zeta(2)$ lies between the tetrahedral reciprocal total and the triangular reciprocal total.

Define the lower and upper gaps

$$A = \zeta(2) - \frac{3}{2} = \frac{\pi^2}{6} - \frac{3}{2},$$

$$B = 2 - \zeta(2) = 2 - \frac{\pi^2}{6}.$$

They satisfy

$$A + B = \frac{1}{2}.$$

The upper gap has the exact triangular-minus-square series

$$\begin{aligned} B &= \sum_{n=1}^{\infty} \left(\frac{1}{T_n} - \frac{1}{n^2} \right) \\ &= \sum_{n=1}^{\infty} \left(\frac{2}{n(n+1)} - \frac{1}{n^2} \right) \\ &= \sum_{n=1}^{\infty} \frac{n-1}{n^2(n+1)} = 2 - \zeta(2). \end{aligned}$$

The lower gap has the tetrahedral-minus-square form

$$\begin{aligned} A &= \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{\binom{n+2}{3}} \right) \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{6}{n(n+1)(n+2)} \right) \\ &= \sum_{n=1}^{\infty} \frac{(n-1)(n-2)}{n^2(n+1)(n+2)}. \end{aligned}$$

The first two terms of the lower-gap series vanish. This gives an elementary explanation for why the upper and lower residues are asymmetric inside the interval $(3/2, 2)$.

Remark 1 (Why this belongs here)**.** *The real-axis bracket is motivational, not causal. It does not locate nontrivial zeros and does not explain the critical-line derivative-gap coupling. Its role is to introduce chamber arithmetic: values above* 1 *are assigned to reciprocal-simplex chambers by a floor of a reciprocal. The same arithmetic form appears again in the annular diagnostic, where the real-axis quantity* $\zeta(s) - 1$ *is replaced by the local isolation product* $P_\rho = |\zeta'(\rho)| h_\rho$*.*

2.3 Real-axis chamber coordinate

For $m \geq 2$, set

$$L_m = \frac{m}{m-1} = 1 + \frac{1}{m-1}.$$

If

$$L_{m+1} < x \leq L_m,$$

then

$$1 + \frac{1}{m} < x \leq 1 + \frac{1}{m-1}.$$

Equivalently,

$$\frac{1}{m} < x - 1 \leq \frac{1}{m-1},$$

so

$$m - 1 \leq \frac{1}{x-1} < m.$$

Thus the chamber index is

$$m_{\mathbb{R}}(x) = \left\lfloor \frac{1}{x-1} \right\rfloor + 1.$$

For $x = \zeta(s)$, $s > 1$, this gives

$$m_{\mathbb{R}}(s) = \left\lfloor \frac{1}{\zeta(s)-1} \right\rfloor + 1.$$

At $s = 2$,

$$\frac{1}{\zeta(2)-1} \approx 1.5505,$$

so

$$m_{\mathbb{R}}(2) = 2,$$

which selects the triangular–tetrahedral chamber

$$L_3 = \frac{3}{2} < \zeta(2) \leq L_2 = 2.$$

This corrected form is structurally parallel to the annular isolation index:

$$m_{\mathbb{R}}(s) = \left\lfloor \frac{1}{\zeta(s)-1} \right\rfloor + 1, \qquad m_{\text{isolate}}(\rho) = \left\lfloor \frac{1}{|\zeta'(\rho)|h_\rho} \right\rfloor + 1.$$

The bridge is not that the real-axis bracket predicts zeros, but that the same reciprocal chamber arithmetic survives the transition from real values to local complex shells.

3 From Real Brackets to Complex Shells

On the critical line, $\zeta(1/2 + it)$ is complex. The real-axis interval bracket no longer applies directly. The replacement is a value-plane shell.

Two natural shell systems are:

$$|\zeta(s)| = \frac{1}{m},$$

centered at the value 0, and

$$|\zeta(s) - 1| = \frac{1}{m},$$

centered at the value 1.

Zeros satisfy $\zeta(\rho) = 0$, so zero-centered shells are the relevant local geometry near nontrivial zeros. By contrast, zeros lie on the boundary

$$|\zeta(\rho) - 1| = 1,$$

so the one-centered shell system describes passages near the value 1, not zero isolation.

4 Zero-Centered Annular Isolation

Definition 1 (Nearest half-gap)**.** *Let*

$$\rho_j = \frac{1}{2} + i\gamma_j$$

be a nontrivial zero on the critical line, with neighboring ordinates γ_{j-1} and γ_{j+1}. Define

$$h_j = \frac{1}{2} \min(\gamma_j - \gamma_{j-1}, \gamma_{j+1} - \gamma_j).$$

Definition 2 (Isolation product and shell index)**.** *For a simple zero ρ_j, define*

$$P_j = \left|\zeta'(\rho_j)\right| h_j,$$

and

$$A_j = \frac{1}{P_j}.$$

The strict isolation index is

$$m_{\text{isolate}}(\rho_j) = \lfloor A_j \rfloor + 1 = \left\lfloor \frac{1}{|\zeta'(\rho_j)|\, h_j} \right\rfloor + 1.$$

Proposition 2 (First-order shell radius)**.** *Assume ρ_j is a simple zero. Near ρ_j,*

$$\zeta(s) = \zeta'(\rho_j)(s - \rho_j) + O((s - \rho_j)^2).$$

Therefore the level set $|\zeta(s)| = 1/m$ has first-order local radius

$$r_m(\rho_j) \approx \frac{1}{m\, |\zeta'(\rho_j)|}.$$

The shell is strictly isolated within the nearest half-gap whenever

$$\frac{1}{m\, |\zeta'(\rho_j)|} < h_j.$$

Equivalently,

$$m > \frac{1}{|\zeta'(\rho_j)|\, h_j}.$$

Thus the strict integer threshold is

$$m_{\text{isolate}}(\rho_j) = \left\lfloor \frac{1}{|\zeta'(\rho_j)|\, h_j} \right\rfloor + 1.$$

Remark 2 (What the diagnostic is and is not)**.** *The diagnostic does not find zeros. It assumes the zero ordinates are already known, then assigns a local isolation coordinate to each zero. It is best understood as a collaring or certification layer: a standard zero locator finds the candidate zero, while the annular-shell diagnostic measures how tightly the $|\zeta|$ levels contract around it relative to neighboring zeros.*

5 Product-Heuristic for the Derivative–Gap Coupling

A heuristic explanation for the observed derivative–gap coupling comes from the product structure of entire functions. At a simple zero ρ_j, the derivative is influenced by the distances from ρ_j to the other zeros. Schematically,

$$|\zeta'(\rho_j)| \approx C(\gamma_j) \prod_{k \neq j} |\rho_j - \rho_k|,$$

where $C(\gamma_j)$ denotes smoother height-dependent factors and contributions outside the local zero neighborhood. Taking logarithms gives

$$\log |\zeta'(\rho_j)| \approx \log C(\gamma_j) + \sum_{k \neq j} \log |\rho_j - \rho_k|.$$

Nearest neighbors therefore contribute directly to the local variation of $|\zeta'(\rho_j)|$. A small adjacent gap suppresses one of the local product factors, and in close-pair cases this suppression combines with the reduced half-gap h_j to produce large values of

$$m_{\text{isolate}}(\rho_j) = \left\lfloor \frac{1}{|\zeta'(\rho_j)| h_j} \right\rfloor + 1.$$

This product heuristic explains why the association is not limited to the most extreme outliers: nearest-neighbor spacing contributes to the derivative scale throughout the sample. The close-pair cases are the visible tail of the same mechanism.

The converse should not be stated pointwise. A small derivative scale need not arise only from a close pair; one-sided imbalance or broader local zero configuration can also contribute. This caveat is consistent with the literature relating close zero pairs of $\zeta(s)$ to zeros of $\zeta'(s)$ near the critical line.

6 Computation and Data Summary

The computations used known zero ordinates from a local zero database, evaluated derivative magnitudes at or near the stored ordinates with `mpmath` at 50 decimal digits, and computed

$$h_j, \quad P_j, \quad A_j, \quad m_{\text{isolate}}(\rho_j).$$

For selected outlier rows, ordinates were refined using a high-precision Hardy Z-function root step, and the isolation metrics were recomputed.

The bulk derivative evaluations are insensitive to ordinary stored-ordinate precision at the scale used here. The rows most vulnerable to numerical artifact are exactly the small-$|\zeta'|$ and small-P outliers, so the refinement pass targeted those rows directly. The refined maxima

$$55 \to 55, \qquad 37 \to 37, \qquad 23 \to 23, \qquad 21 \to 21$$

confirm that the reported refined maxima are not float64 artifacts.

The audit proceeded in four stages:

1. a baseline low block of 4519 zeros;
2. a disjoint block of 1000 zeros immediately above the baseline block;
3. three high-altitude sentinel bands of 250 zeros near $T = 10^5$, 10^6, and $9 \cdot 10^6$;
4. permutation tests shuffling derivative magnitudes against local half-gaps.

7 Baseline and Disjoint-Block Results

Table 1 summarizes the baseline and disjoint-block audits.

Block	Zeros	$\gamma_{\min}$	$\gamma_{\max}$	med m	q_{95}	q_{99}	max m	Spearman
Low baseline	4519	14.13	4998.37	1	3	8	55	0.598
Disjoint block	1000	5000.83	5928.90	1	4	8	105	0.788

Table 1: Baseline and disjoint-block shell-isolation statistics. Quantiles of the integer-valued isolation index are reported as rounded chamber counts. The disjoint block replicated the derivative-gap coupling and produced a larger finite-range maximum.

The baseline block had

$$\text{median}(m_{\text{isolate}}) = 1, \qquad q_{95} = 3, \qquad q_{99} = 8, \qquad \max_{\text{tested}} = 55.$$

The disjoint block had

$$\text{median}(m_{\text{isolate}}) = 1, \qquad q_{95} = 4, \qquad q_{99} \approx 8, \qquad \max_{\text{tested}} = 105.$$

The increase from 55 to 105 confirms that the baseline maximum should not be interpreted as a bound.

7.1 Fixed-shell overlap audit

For the baseline block, fixed shells gave the following overlap counts, where an overlap means

$$r_m(\rho_j) > h_j.$$

m	overlap count	overlap fraction	max radius / half-gap
2	396/4519	0.0876	27.375
3	185/4519	0.0409	18.250
5	78/4519	0.0173	10.950
10	28/4519	0.0062	5.475
100	0/4519	0	0.548
1000	0/4519	0	0.0548
10000	0/4519	0	0.00548

Table 2: Baseline fixed-shell overlap audit. Shell $m = 100$ isolated all baseline zeros in this finite range.

8 Double-Penalty Outliers

The strongest isolation outliers are dominated by adjacent close-zero pairs. In the baseline top-50 isolation outliers, 46/50 rows belonged to adjacent-pair events. The outliers show a simultaneous suppression of both factors in

$$P_j = \left|\zeta'(\rho_j)\right| h_j.$$

The baseline top-50 medians compared with the non-top population were:

$$\text{median}_{\text{top50}} \left|\zeta'\right| = 1.02879, \qquad \text{median}_{\text{non-top}} \left|\zeta'\right| = 4.35022,$$

so the derivative scale was about 4.23 times smaller in the top-50 outliers. Likewise,

$$\text{median}_{\text{top50}}\, h = 0.0912402, \qquad \text{median}_{\text{non-top}}\, h = 0.394396,$$

so the local half-gap was about 4.32 times smaller. Consequently,

$$\text{median}_{\text{top50}}\, P = 0.0962339, \qquad \text{median}_{\text{non-top}}\, P = 1.74112,$$

an approximately 18.09-fold reduction in the product.

The worst baseline pair occurred near

$$\gamma \approx 4292.726445, \qquad 4292.817263,$$

with gap approximately 0.090818, half-gap approximately 0.045409, and derivative magnitudes about 0.402 and 0.408, producing

$$m_{\text{isolate}} = 55, \qquad 54.$$

The worst disjoint-block event occurred near

$$\gamma \approx 5229.198557, \qquad 5229.241811,$$

with gap approximately 0.043254, half-gap approximately 0.021627, and derivative magnitudes about 0.445 and 0.454, producing

$$m_{\text{isolate}} = 105, \qquad 102.$$

9 Correlation, Height Control, and Permutation Audits

In the baseline block,

$$\text{Spearman}\big(\left|\zeta'(\rho_j)\right|, h_j\big) = 0.598107.$$

After removing the top-50 outliers, the correlation remained

$$0.584583.$$

After excluding the top 5% of isolation outliers, it remained

$$0.547902.$$

Thus the derivative-gap association is not only a top-tail effect.

The wide baseline block mixes local geometry with height drift. The average zero spacing changes with height, approximately on the scale $2\pi/\log(\gamma/2\pi)$, while the typical size and distribution of $|\zeta'(\rho)|$ also vary with height. These effects make narrow-band checks important. The high-altitude sentinel bands are effectively height-controlled compared with the baseline block, and they show even stronger within-band rank correlations:

$$0.7398, \qquad 0.6723, \qquad 0.6789.$$

This supports the interpretation that the raw baseline value 0.598107 is not produced by a height confound. If anything, the narrow-band results suggest that the local coupling is clearer after height drift is reduced.

For reference, a local normalized half-gap can be written as

$$\widetilde{h}_j = h_j \frac{\log(\gamma_j/2\pi)}{\pi},$$

because $\pi/\log(\gamma_j/2\pi)$ is the approximate local mean half-spacing. A full unfolded analysis could combine this normalization with a height-detrended derivative scale, but the sentinel-band correlations already provide a direct height-controlled audit.

A permutation audit shuffled the observed derivative magnitudes against the observed half-gaps. In the baseline block, the minimum-P test produced 2 hits in 2000 permutations, while the top-k median-P statistic produced 0 hits in 2000. In the disjoint block, the minimum-P test produced 3 hits in 2000, while the top-k median-P statistic again produced 0 hits in 2000.

The top-k statistic is the more robust tail diagnostic: a single extreme event can sometimes be reproduced by random pairing, but the collective top-tail suppression of P_j was not reproduced in these tests.

10 High-Altitude Sentinel Bands

To test whether the diagnostic survived larger heights, three sentinel bands of 250 zeros each were sampled near

$$T = 10^5, \qquad 10^6, \qquad 9 \cdot 10^6.$$

The results are shown in Table 3.

Band	Zeros	med m	q_{95}	q_{99}	max m	med P	$q_{01}(P)$	Spearman
$T \sim 10^5$	250	1	5	7	37	1.5898	0.1549	0.7398
$T \sim 10^6$	250	1	5	10	23	1.6536	0.1086	0.6723
$T \sim 9 \cdot 10^6$	250	1	4	16	21	1.6327	0.0670	0.6789

Table 3: High-altitude sentinel bands. Quantiles of m_{isolate} are rounded chamber counts. The finite-range maxima vary by sampled band; the stable feature is the positive derivative-gap rank correlation.

The maxima in the sentinel bands,

$$37, \quad 23, \quad 21,$$

were all stable under top-outlier refinement:

$$37 \to 37, \qquad 23 \to 23, \qquad 21 \to 21.$$

The top-20 outlier rows in the sentinel bands remained pair-dominated:

$$16/20, \quad 16/20, \quad 14/20$$

paired rows, respectively.

The sentinel results should not be read as an altitude law for the maximum isolation index. Each band contains only 250 zeros, so the maximum is an extreme-value statistic. The robust altitude conclusion is that the positive derivative-gap coupling persisted through the $9 \cdot 10^6$ stress band.

The 1% lower tail of the product also moved downward across the sentinel bands:

$$q_{01}(P): \quad 0.1549o0.1086o0.0670.$$

Since m_{isolate} is governed by $1/P$, this shrinking small-P tail is consistent with the expectation that larger isolation events can appear at greater heights. The sample sizes are small, so this should be read only as weak finite-range evidence, not as an asymptotic law.

11 Literature Positioning

The two ingredients of P_j are classical objects. Zero spacings are connected to Montgomery's pair-correlation conjecture and the GUE model of local statistics [1, 2, 3, 4]. Derivatives at zeros and their moments have been studied by Gonek, Hejhal, Hughes–Keating–O'Connell, Ng, Milinovich, and others [5, 6, 7, 8, 9]. Results on the typical size and distribution of $\log |\zeta'(ho)|$ also motivate height-controlled comparisons [**?**].

The observed close-pair/small-derivative behavior is consistent with existing theory. Zhang, Ki, Farmer–Ki, and Radziwiłł connect close zero pairs of $\zeta(s)$ with zeros of $\zeta'(s)$ near the critical line [10, 11, 12, 13]. Random-matrix analogues for characteristic polynomials and their derivatives were studied by Mezzadri and by Dueñez–Farmer–Froehlich–Hughes–Mezzadri–Phan [14, 15]. Simplicity questions and derivative moments also appear in the work of Conrey–Ghosh–Gonek [**?**].

Thus the annular-shell construction is best described as a finite-range diagnostic and visualization coordinate for known local zero geometry. The product P_j is not claimed as a new invariant. Its value is that it packages derivative scale and local spacing into a single isolation coordinate.

12 Limitations

Several limitations are essential.

First, all reported maxima are finite-range tested maxima. They are not bounds. In fact, based on the expected existence of arbitrarily small normalized gaps and arbitrarily small values of $|\zeta'(\rho)|$, the reciprocal product

$$\frac{1}{P_j} = \frac{1}{|\zeta'(\rho_j)|\, h_j}$$

should not be expected to remain bounded globally.

Second, the diagnostic assumes known zero ordinates. It does not locate zeros by itself. A zero-finding pipeline would require a locator such as Hardy Z-function sign changes, Gram point bracketing, Riemann–Siegel evaluation, Newton refinement, or argument-principle counting. The annular-shell diagnostic can then be applied as a local isolation certificate.

Third, the linearized radius formula is first-order. It is accurate only sufficiently near simple zeros. The diagnostic is therefore a local approximation, not a global description of level curves of $\zeta(s)$.

Fourth, the visible value-plane plots of $\zeta(1/2 + it)$ may show angular or polygonal-looking paths under sparse sampling. Such features should be treated as visualization artifacts unless tested by denser sampling and scatter-only plots.

13 Conclusion

The triangular–tetrahedral bracket

$$\frac{3}{2} < \zeta(2) < 2$$

motivated a shift from real-axis reciprocal-simplex intervals to complex value-plane shells. On the critical line, the zero-centered shell system gives a local isolation coordinate around known zeros:

$$m_{\mathrm{isolate}}(\rho_j) = \left\lfloor \frac{1}{|\zeta'(\rho_j)|\, h_j} \right\rfloor + 1.$$

Empirically, across a baseline low block, a disjoint block, and high-altitude sentinel bands up to $T \sim 9 \cdot 10^6$, the product

$$P_j = \left|\zeta'(\rho_j)\right| h_j$$

provides a stable finite-range coordinate for shell isolation. The largest isolation events are close-pair dominated, and the derivative-gap rank correlation persists across sampled heights. The construction supplies a useful diagnostic lens on known local zero geometry, while making no claim of a new invariant, no global bound, and no implication for the Riemann Hypothesis.

A Reproducibility Ledger

The main numerical settings were:

- Working precision: 50 decimal digits for derivative evaluations and refinement checks.
- Baseline block: 4519 zeros, $\gamma \approx 14.13$ to 4998.37.
- Disjoint block: 1000 zeros, $\gamma \approx 5000.83$ to 5928.90.
- Sentinel bands: 250 zeros each near 10^5, 10^6, and $9 \cdot 10^6$.
- Permutation tests: 2000 shuffles per block or band.
- Outlier refinement: baseline top 100 plus random controls; sentinel top 20 per band.

The key finite-range maxima were:

$$55, \quad 105, \quad 37, \quad 23, \quad 21.$$

The refined maxima available in the audit were:

$$55 \to 55, \qquad 37 \to 37, \qquad 23 \to 23, \qquad 21 \to 21.$$

B Safe Claim Checklist

The following statements are safe:

- The annular-shell diagnostic reduces first-order local isolation to $P_j = |\zeta'(\rho_j)|\, h_j$.
- The derivative magnitude and local half-gap are positively rank-correlated in all sampled blocks.
- The strongest isolation outliers are close-pair dominated.
- High-precision refinement confirms that reported refined maxima are not float64 artifacts.
- The results are finite-range empirical diagnostics.

The following statements should be avoided:

- that m_{isolate} is globally bounded;
- that the diagnostic proves or suggests the Riemann Hypothesis;
- that P_j is a new invariant of the zeta function;
- that the triangular–tetrahedral bracket causes the zero-shell behavior;
- that value-plane polygonal traces are structural without denser sampling checks.

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